JOURNAL OF ALGEBRA **198,** 135–163 (1997)
ARTICLE NO. JA977159

Approximations and Almost Split Sequences in Homologically Finite Subcategories*

Mark Kleiner

Department of Mathematics, Syracuse University, Syracuse, New York 13244-1150

Communicated by Kent R. Fuller

Received August 6, 1996

1. INTRODUCTION

In this paper, Λ is an artin algebra over a commutative artinian ring R, the radical of Λ is denoted by $\mathbf{r}$, and $\operatorname{mod}\Lambda$ is the category of finitely generated left Λ-modules. The definitions, notation, and results of [ARS] on rings, modules, and artin algebras are used freely throughout the paper, often without an explicit reference.

We remind the reader of some of the definitions and notation of [AS80, AS81a] concerning subcategories of $\operatorname{mod}\Lambda$. By a subcategory, we always mean a full subcategory closed under isomorphisms.

Throughout the paper, $\mathcal{X}$ is a subcategory of $\operatorname{mod}\Lambda$ closed under direct summands and extensions, the latter meaning that if $0 \to A \to B \to C \to 0$ is an exact sequence in $\operatorname{mod}\Lambda$ and $A, C \in \mathcal{X}$, then $B \in \mathcal{X}$. An exact sequence in $\mathcal{X}$ is an exact sequence in $\operatorname{mod}\Lambda$ with all the terms in $\mathcal{X}$. A module $Y \in \mathcal{X}$ is said to be Ext-injective if $\operatorname{Ext}^1_\Lambda(X, Y) = 0$ for all $X \in \mathcal{X}$; Y is splitting injective if every monomorphism $Y \to X$ with $X \in \mathcal{X}$ splits. A splitting injective module is Ext-injective, but the converse, generally speaking, is not true. The Ext-projective modules are defined dually.

A morphism $f: A \to B$ in $\mathcal{X}$ is a left almost split morphism in $\mathcal{X}$ if it is not a split monomorphism and every morphism $j: A \to X$ in $\mathcal{X}$ that is not

* The main results of this paper were obtained in April–May 1996, when the author visited Instituto de Matemáticas, Universidad Nacional Autónoma de México, and was partially supported by project IN103195-DGAPA UNAM. The author is grateful to Raymundo Bautista, Roberto Martínez-Villa, and José Antonio de la Peña for several stimulating mathematical conversations. E-mail address: mkleiner@sound.syr.edu.

0021-8693/97 $25.00

a split monomorphism factors through f. A right almost split morphism in $\mathcal{X}$ is defined by duality. An exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ in $\mathcal{X}$ is said to be an almost split sequence in $\mathcal{X}$ if f is a left almost split morphism in $\mathcal{X}$ and g is a right almost split morphism in $\mathcal{X}$.

Given a module $Z \in \operatorname{mod} \Lambda$, a morphism $f : Z \to X(Z)$ with $X(Z) \in \mathcal{X}$ is said to be a left $\mathcal{X}$-approximation of Z if for every morphism $j : Z \to X$ with $X \in \mathcal{X}$, there exists a morphism $h : X(Z) \to X$ satisfying $j = hf$. A left $\mathcal{X}$-approximation f is called a minimal left $\mathcal{X}$-approximation if f is a left minimal morphism, i.e., if every endomorphism $s : X(Z) \to X(Z)$ satisfying $sf = f$ is an isomorphism. A minimal left $\mathcal{X}$-approximation is unique up to isomorphism. A module has a left $\mathcal{X}$-approximation if and only if it has a minimal left $\mathcal{X}$-approximation. The subcategory $\mathcal{X}$ is said to be covariantly finite if every module in $\operatorname{mod} \Lambda$ has a left $\mathcal{X}$-approximation.

The notions of a right or minimal right $\mathcal{X}$-approximation, as well as of a contravariantly finite subcategory of $\operatorname{mod} \Lambda$ are introduced by duality. $\mathcal{X}$ is said to be functorially finite if it is both covariantly and contravariantly finite. The categories that are either covariantly or contravariantly finite are referred to as homologically finite [AR92].

An important role in the theory of almost split sequences in $\operatorname{mod} \Lambda$ is played by the fact that if $0 \to A \to B \to C \to 0$ is an almost split sequence, then $A \cong \mathrm{DTr}\, C$ and $C \cong \mathrm{TrD}\, A$, where $\mathrm{D} : \operatorname{mod} \Lambda \to \operatorname{mod} \Lambda^{op}$ and $\mathrm{Tr} : \operatorname{mod} \Lambda \to \operatorname{mod} \Lambda^{op}$ are the well-known duality and operator, respectively. In particular, the most elegant known proof of existence of almost split sequences is based on the application of D and Tr to the study of the covariant and contravariant defect functors of an exact sequence in $\operatorname{mod} \Lambda$ [ARS].

The general theory of almost split sequences was further developed in [AS81a], where for $\mathcal{X}$ functorially finite, the existence of almost split sequences in $\mathcal{X}$ was established. The authors of [AS81a] proposed the following problems (p. 435).

(a) Describe the Ext-projective and Ext-injective objects in $\mathcal{X}$. In particular, need there be only a finite number of isomorphism classes of indecomposable objects which are Ext-projective or Ext-injective? Also, need these numbers be the same when finite?

(b) Suppose $0 \to A \to B \to C \to 0$ is an almost split sequence in $\mathcal{X}$. Does there exist a method of constructing A from C and vice versa similar to the construction given by the functors DTr and TrD?

In [AS81a] these problems were not solved in general, but information along these lines was presented in some special situations. In the years that followed, both problems have been addressed in numerous special cases. However, significant general progress has been achieved only on problem (a) in the case when there is a connection with the tilting theory [AR91a, AR92]. These developments were inspired by [AS81b].

In the present paper, we solve problem (b) in terms of minimal left or right $\mathcal{X}$-approximations, DTr, TrD, and functor Ext^1_Λ. We also solve that part of problem (a) that concerns the description of Ext-projective and Ext-injective modules. Our description gives a necessary and sufficient condition for the finiteness of number of pairwise non-isomorphic indecomposable Ext-projective or Ext-injective modules in terms of the minimal $\mathcal{X}$-approximation of a certain subcategory associated with $\mathcal{X}$.

Our results on problems (a) and (b) hold when $\mathcal{X}$ is either covariantly or contravariantly finite, which is a weaker assumption than that of $\mathcal{X}$ being functorially finite. In particular, based on the application of left or right $\mathcal{X}$-approximations to the study of the contravariant or covariant defect functor, respectively, of a short exact sequence in $\mathcal{X}$, we obtain two "one-sided" existence theorems for almost split sequences in $\mathcal{X}$ and, combining both "one-sided" theorems, a new proof of the main existence theorem of [AS81a, Theorem 2.4, p. 434]. The new proof does not depend on the theory of dualizing R-varieties.

The following statement (Lemma 2.1) serves as a crucial technical tool. A minimal left $\mathcal{X}$-approximation $f^Z : Z \to X^Z$ of Z induces a monomorphism $\mathrm{Ext}^1_\Lambda(f^Z, \ ) : \mathrm{Ext}^1_\Lambda(X^Z, \ )|\mathcal{X} \to \mathrm{Ext}^1_\Lambda(Z, \ )|\mathcal{X}$ of the indicated restrictions of functor Ext^1_Λ to $\mathcal{X}$. A corollary is Wakamatsu's Lemma [W], which plays an important role in the theory of homologically finite subcategories closed under extensions [AR91a, AR92]. Lemma 2.1 is presented in Section 2, where it is used to prove the following result (Theorem 2.3). If $\mathcal{X}$ is covariantly finite and A is an indecomposable non-Ext-injective module in $\mathcal{X}$, then the minimal left approximation $X^{\mathrm{TrD}\,A}$ of $\mathrm{TrD}\,A$ can be written in the form $X^{\mathrm{TrD}\,A} = C \oplus C'$, where $\mathrm{Ext}^1_\Lambda(C, A) \neq 0$, $\mathrm{Ext}^1_\Lambda(C', A) = 0$, and $C = \sigma A$ is a (uniquely determined) indecomposable module. The rest of the section contains a proof of the existence of an almost split sequence $0 \to A \to E \to \sigma A \to 0$ in $\mathcal{X}$ by extending the classical techniques of [ARS], including the analysis of covariant and contravariant defects of an exact sequence in $\mathcal{X}$.

As we have just indicated, if A is the left end-term of an almost split sequence in $\mathcal{X}$, then the right end-term σA is a direct summand of $X^{\mathrm{TrD}\,A}$, a minimal left $\mathcal{X}$-approximation of $\mathrm{TrD}\,A$. It is an interesting problem to describe those subcategories $\mathcal{X}$ for which $X^{\mathrm{TrD}\,A}$ is indecomposable for all indecomposable non-Ext-injective modules $A \in \mathcal{X}$ and, hence, $\sigma A \cong X^{\mathrm{TrD}\,A}$. We give examples of such subcategories in Section 4.

The dual results on right $\mathcal{X}$-approximations and contravariantly finite subcategories $\mathcal{X}$ are stated for the convenience of the reader.

Section 3 contains various characterizations of Ext-injective and Ext-projective modules in $\mathcal{X}$. For example, if $\mathcal{X}$ is covariantly finite, then an indecomposable module $M \in \mathcal{X}$ is Ext-injective if and only if $\operatorname{Ext}^1_\Lambda(X^{\operatorname{TrD} M}, M) = 0$, and M is Ext-projective if and only if it is either a splitting projective module or a direct summand of the cokernel of a minimal left $\mathcal{X}$-approximation of a module in $\operatorname{mod} \Lambda$. The utility of the first of these two results is that in order to verify whether M is Ext-injective, instead of checking whether $\operatorname{Ext}^1_\Lambda(X, M) = 0$ for all $X \in \mathcal{X}$, it suffices to check whether $\operatorname{Ext}^1_\Lambda(X, M) = 0$ for one only value of X, namely, $X = X^{\operatorname{TrD} M}$. The second result gives a description of the full subcategory of $\operatorname{mod} \Lambda$ determined by the Ext-projective modules of $\mathcal{X}$. Combining these results with their duals, we obtain several equivalent descriptions of Ext-injective or Ext-projective modules in case $\mathcal{X}$ is functorially finite. In addition, this section contains the aforementioned necessary and sufficient condition for the finiteness of number of non-isomorphic indecomposable Ext-injective (Ext-projective) modules, as well as a sufficient condition for this finiteness if $\mathcal{X}$ is a homologically finite subcategory associated with a pair of adjoint functors.

In Section 3, we also construct the following curious short exact sequence in $\operatorname{mod} \Lambda$. Suppose $\mathcal{X}$ is covariantly finite, $L \in \mathcal{X}$ is an indecomposable Ext-injective but not splitting injective module, and $f : L \to M$ is a minimal left almost split morphism in $\mathcal{X}$. Then the sequence $0 \to L \xrightarrow{f} M \xrightarrow{g} \operatorname{Coker} f \to 0$ is exact, $\operatorname{Coker} f \notin \mathcal{X}$ is indecomposable, and g is a minimal right $\mathcal{X}$-approximation of $\operatorname{Coker} f$. Clearly, the sequence is not an almost split sequence in $\mathcal{X}$, but it behaves very much like one. For every morphism $X \to \operatorname{Coker} f$ with $X \in \mathcal{X}$ that is not a split epimorphism, i.e., every such morphism because $\operatorname{Coker} f \notin \mathcal{X}$ is indecomposable, can be lifted to M.

Section 4 contains applications of the general theory developed in Sections 2 and 3 to subcategories closed under submodules. The dual consideration of subcategories closed under factor modules is left to the reader.

We first show that some of the results of [AS81a] on $\mathcal{X}$ closed under submodules are consequences of our general results. For instance, Proposition 3.1(c) of [AS81a, p. 436] follows immediately from the description of Ext-projective modules, and Corollaries 3.4 and 3.5 of [AS81a, pp. 437−438] are special cases of the description of Ext-injective modules and of the existence of almost split sequences, respectively, in an arbitrary covariantly finite $\mathcal{X}$. Then we consider the case $\mathcal{X} = \operatorname{Sub} M$ with $M \in \operatorname{mod} \Lambda$. Recall that $\operatorname{Sub} M$ is the subcategory of $\operatorname{mod} \Lambda$ consisting of all modules isomorphic to submodules of finite direct sums of copies of M; sufficient conditions for $\operatorname{Sub} M$ to be closed under extensions are given in Section 5 of [AS81a]. For $\mathcal{X} = \operatorname{Sub} M$, we give a shorter proof of the description obtained in [AS81a] of Ext-injective but not splitting injective modules.

Then we concentrate on the case $\mathcal{X} = \operatorname{Sub}\Lambda$, which was studied in [BM] for a 1-Gorenstein artin algebra Λ. According to the necessary and sufficient condition of [AS81a], the class of artin algebras Λ with $\operatorname{Sub}\Lambda$ closed under extensions is strictly larger than the class of 1-Gorenstein algebras. So a natural question is whether the formulas of [BM] for left and right end-terms apply to an almost split sequence $0 \to A \to B \to C \to 0$ in an arbitrary subcategory $\operatorname{Sub}\Lambda$ closed under extensions. We show that the formula $C \cong \Omega \operatorname{Tr} \Omega \operatorname{D} A$, given in [BM] for a 1-Gorenstein algebra Λ, works in general; here ΩN denotes the first syzygy of the module N. On the other hand, we show by example that the formula $A \cong \Omega \operatorname{D} \Omega \operatorname{Tr} C$ of [BM] does not work in general. The general formula is $A \cong (\Omega \operatorname{D} \Omega \operatorname{Tr} C)_{\Omega(\mathcal{I})}$, where for a module $L \in \operatorname{mod}\Lambda$, the module $L_{\Omega(\mathcal{I})}$ is obtained from L by dropping all those indecomposable direct summands of L that are also direct summands of ΩI for some indecomposable injective Λ-module I. Equivalently, $L_{\Omega(\mathcal{I})}$ is obtained from L by dropping all indecomposable Ext-injective but not splitting injective direct summands of L. It would be interesting to describe those artin algebras Λ for which $\operatorname{Sub}\Lambda$ is closed under extensions and $A \cong \Omega \operatorname{D} \Omega \operatorname{Tr} C$ for all almost split sequences in $\operatorname{Sub}\Lambda$. We give an example of a non-1-Gorenstein artin algebra having these properties. In the situation when $\operatorname{inj\,dim}_\Lambda \Lambda = 1$, the formula for the left end-term of an almost split sequence in the category of Cohen−Macaulay modules over a Gorenstein artin algebra Λ [AR 91b, Theorem 3.7, p. 234] is another special case of our formula $A \cong (\Omega \operatorname{D} \Omega \operatorname{Tr} C)_{\Omega(\mathcal{I})}$.

In Section 4, we also give an explicit formula for a right (not necessarily minimal) $\mathcal{X}$-approximation of an arbitrary module in $\operatorname{mod}\Lambda$ if $\mathcal{X} = \operatorname{Sub}\Lambda$. While an explicit formula for a minimal left $\mathcal{X}$-approximation of an arbitrary module is given in [AS81a] for any $\mathcal{X}$ closed under submodules, it seems that no explicit formula for a right $\operatorname{Sub}\Lambda$-approximation has appeared in the literature.

The author is grateful for the helpful remarks of the referee.

2. LEFT APPROXIMATIONS AND THE RIGHT END-TERM OF AN ALMOST SPLIT SEQUENCE

In what follows, for all $Z \in \operatorname{mod}\Lambda$, we denote by $f^Z : Z \to X^Z$ a minimal left $\mathcal{X}$-approximation of Z (if it exists), and by $g_Z : X_Z \to Z$, a minimal right $\mathcal{X}$-approximation of Z.

We begin by showing that a minimal left or right $\mathcal{X}$-approximation induces a monomorphism of the appropriate restrictions of functor $\operatorname{Ext}^1_\Lambda$ to $\mathcal{X}$; this result is crucial for the rest of the paper.

LEMMA 2.1. *For all $Z \in \operatorname{mod} \Lambda$:*

(a) $\operatorname{Ext}^1_\Lambda(f^Z, \ ) : \operatorname{Ext}^1_\Lambda(X^Z, \ )|\mathsf{X} \to \operatorname{Ext}^1_\Lambda(Z, \ )|\mathsf{X}$ *is a monomorphism of functors.*

(b) $\operatorname{Ext}^1_\Lambda(\ , g_Z) : \operatorname{Ext}^1_\Lambda(\ , X_Z)|\mathsf{X} \to \operatorname{Ext}^1_\Lambda(\ , Z)|\mathsf{X}$ *is a monomorphism of contravariant functors.*

Proof. (a) Consider the exact commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r] \arrow[d, equal] & V \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d, "f^Z"] & 0 \\ 0 \arrow[r] & N \arrow[r] & W \arrow[r, "g"] & X^Z \arrow[r] & 0 \end{tikzcd}$$

in $\operatorname{mod} \Lambda$ with $N \in \mathsf{X}$, and suppose that the top row splits. Then $f^Z = gt$ for some $t : Z \to W$ and, since X is closed under extensions, $W \in \mathsf{X}$ so that $t = hf^Z$ for some $h : X^Z \to W$. We obtain $f^Z = ghf^Z$, whence gh is an isomorphism because f^Z is left minimal, and the bottom row splits. This shows that $\operatorname{Ext}^1_\Lambda(f^Z, N) : \operatorname{Ext}^1_\Lambda(X^Z, N) \to \operatorname{Ext}^1_\Lambda(Z, N)$ is a monomorphism of R-modules.

(b) This is dual to the proof of (a). ∎

The next statement was proved in [W] (see Proposition 2.22, parts (2) and (2*), pp. 318–319) in a special case, then stated in full generality, called Wakamatsu's Lemma, and used extensively in [AR91a, AR92], where it was recognized that the original proof works in general. We obtain the statement as a consequence of Lemma 2.1.

COROLLARY 2.2 (Wakamatsu's Lemma). *For all $Z \in \operatorname{mod} \Lambda$:*

(a) $\operatorname{Ext}^1_\Lambda(\operatorname{Coker} f^Z, \ ) \,|\, \mathsf{X} = 0$.

(b) $\operatorname{Ext}^1_\Lambda(\ , \operatorname{Ker} g_Z) \,|\, \mathsf{X} = 0$.

Proof. (a) Let $j : X^Z \to \operatorname{Coker} f^Z$ be the natural projection, then $\operatorname{Ext}^1_\Lambda(f^Z, \ ) \circ \operatorname{Ext}^1_\Lambda(j, \ ) = 0$, whence $\operatorname{Ext}^1_\Lambda(j, \ ) \,|\, \mathsf{X} = 0$ by Lemma 2.1(a). For any $N \in \mathsf{X}$, applying the contravariant functor $\operatorname{Hom}_\Lambda(\ , N)$ to the short exact sequence $0 \to \operatorname{Im} f^Z \xrightarrow{i} X^Z \xrightarrow{j} \operatorname{Coker} f^Z \to 0$, we obtain the long exact sequence $\cdots \to \operatorname{Hom}_\Lambda(X^Z, N) \xrightarrow{\operatorname{Hom}_\Lambda(i, N)} \operatorname{Hom}_\Lambda(\operatorname{Im} f^Z, N) \xrightarrow{\sigma} \operatorname{Ext}^1_\Lambda(\operatorname{Coker} f^Z, N) \xrightarrow{\operatorname{Ext}^1_\Lambda(j, N)} \operatorname{Ext}^1_\Lambda(X^Z, N) \cdots$ in $\operatorname{mod} R$. By the above remark, $\operatorname{Ext}^1_\Lambda(j, N) = 0$. Clearly, $i : \operatorname{Im} f^Z \to X^Z$ is a left X-approximation of $\operatorname{Im} f^Z$, so that $\operatorname{Hom}_\Lambda(i, N)$ is onto and, by the exactness, $\sigma = 0$. It follows that $\operatorname{Ext}^1_\Lambda(\operatorname{Coker} f^Z, N) = 0$.

(b) This is dual to the proof of (a). ∎

If f^Z is a monomorphism (g_Z is an epimorphism), it is easy to see that Wakamatsu's Lemma implies Lemma 2.1. We do not know whether the two lemmas are equivalent in general.

We are ready to present a method for computing one of the end-terms of an almost split sequence in a homologically finite subcategory $\mathcal{X}$ if the other end-term is given.

THEOREM 2.3. (a) *Let $C \in \mathcal{X}$ be an indecomposable non-Ext-projective module. If* $\mathrm{DTr}\, C$ *has a right $\mathcal{X}$-approximation, then $X_{\mathrm{DTr}\, C} = A \oplus A'$, where A is an indecomposable module,* $\mathrm{Ext}^1_\Lambda(C, A) \neq 0$, *and* $\mathrm{Ext}^1_\Lambda(C, A') = 0$. *The module A is determined uniquely up to isomorphism.*

(b) *In the setting of* (a), *denote by $\sigma\tau C$ the unique indecomposable direct summand of $X_{\mathrm{DTr}\, C}$ satisfying* $\mathrm{Ext}^1_\Lambda(C, \tau C) \neq 0$. *A non-split short exact sequence* $0 \to X_{\mathrm{DTr}\, C} \to E \to C \to 0$ *is isomorphic to the direct sum of a split exact sequence* $0 \to Y \to Y \to 0 \to 0$ *and a non-split exact sequence* $0 \to \tau C \to U \to C \to 0$.

(c) *Let $A \in \mathcal{X}$ be an indecomposable non-Ext-injective module. If* $\mathrm{TrD}\, A$ *has a left $\mathcal{X}$-approximation, then $X^{\mathrm{TrD}\, A} = C \oplus C'$, where C is an indecomposable module,* $\mathrm{Ext}^1_\Lambda(C, A) \neq 0$, *and* $\mathrm{Ext}^1_\Lambda(C', A) = 0$. *The module C is determined uniquely up to isomorphism.*

(d) *In the setting of* (c), *denote by σA the unique indecomposable direct summand of $X^{\mathrm{TrD}\, A}$ satisfying* $\mathrm{Ext}^1_\Lambda(\sigma A, A) \neq 0$. *A non-split short exact sequence* $0 \to A \to F \to X^{\mathrm{TrD}\, A} \to 0$ *is isomorphic to the direct sum of a split exact sequence* $0 \to 0 \to W \to W \to 0$ *and a non-split exact sequence* $0 \to A \to V \to \sigma A \to 0$.

Proof. (a) Write $X_{\mathrm{DTr}\, C} = A_1 \oplus \cdots \oplus A_m$, where A_j is indecomposable for all $j = 1, \dots, m$. By Lemma 2.1(b), $\mathrm{Ext}^1_\Lambda(C, g_{\mathrm{DTr}\, C}) : \mathrm{Ext}^1_\Lambda(C, X_{\mathrm{DTr}\, C}) \to \mathrm{Ext}^1_\Lambda(C, \mathrm{DTr}\, C)$ is a monomorphism of R-modules which, in fact, is a monomorphism of $\mathrm{End}_\Lambda(C)^{op}$-modules in view of Lemma III.1.6 of [M]. Therefore, $\mathrm{Im}\,\mathrm{Ext}^1_\Lambda(C, g_{\mathrm{DTr}\, C})$ is an $\mathrm{End}_\Lambda(C)^{op}$-submodule of $\mathrm{Ext}^1_\Lambda(C, \mathrm{DTr}\, C)$ isomorphic to $\mathrm{Ext}^1_\Lambda(C, X_{\mathrm{DTr}\, C}) \cong \bigoplus_{j=1}^m \mathrm{Ext}^1_\Lambda(C, A_j)$, using the fact that $\mathrm{Ext}^1_\Lambda(C, \ ) : \mathrm{mod}\, \Lambda \to \mathrm{mod}\, \mathrm{End}_\Lambda(C)^{op}$ is an additive functor. Since C is not Ext-projective in $\mathcal{X}$, it is not projective in $\mathrm{mod}\, \Lambda$, so that $\mathrm{Ext}^1_\Lambda(C, \mathrm{DTr}\, C)$ has a simple socle according to V Proposition 2.1 of [ARS]. It follows that $\mathrm{Im}\,\mathrm{Ext}^1_\Lambda(C, g_{\mathrm{DTr}\, C})$ is either zero or an indecomposable $\mathrm{End}_\Lambda(C)^{op}$-module, whence there is at most one j satisfying $\mathrm{Ext}^1_\Lambda(C, A_j) \neq 0$.

To finish the proof of (a), it suffices to show that $\mathrm{Ext}^1_\Lambda(C, X_{\mathrm{DTr}\, C}) \neq 0$. Since C is not Ext-projective in $\mathcal{X}$, there exists a non-split exact sequence

$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ in $\mathcal{X}$, and we obtain the exact commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "j"'] & B \arrow[r, "g"] \arrow[d] & C \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathrm{DTr}\, C \arrow[r] & F \arrow[r] & C \arrow[r] & 0,
\end{tikzcd}$$

where the bottom row is an almost split sequence in mod Λ. Since $A \in \mathcal{X}$, we have $j = g_{\mathrm{DTr}\, C} h$ for some $h : A \to X_{\mathrm{DTr}\, C}$, which leads to the following exact commutative diagram in mod Λ.

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "h"'] & B \arrow[r, "g"] \arrow[d, "s"] & C \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & X_{\mathrm{DTr}\, C} \arrow[r] \arrow[d, "g_{\mathrm{DTr}\, C}"'] & E \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathrm{DTr}\, C \arrow[r] & F \arrow[r] & C \arrow[r] & 0.
\end{tikzcd} \tag{2.1}$$

Since the bottom row does not split, neither does the middle row, whence $\mathrm{Ext}^1_\Lambda(C, X_{\mathrm{DTr}\, C}) \neq 0$.

(b) This follows from (a), properties of the Baer addition of short exact sequences, and the additivity of the functor $\mathrm{Ext}^1_\Lambda(C, \ )$.

(c) and (d) These are dual to the proofs of (a) and (b). ∎

In view of Theorem 2.3, we introduce the following notation.

Notation 2.1. Let $M \in \mathcal{X}$ be an indecomposable module.

(a) Suppose $\mathcal{X}$ is contravariantly finite. We put $\tau M = 0$ if M is Ext-projective. If M is not Ext-projective, then τM is the unique indecomposable direct summand Y of $X_{\mathrm{DTr}\, M}$ satisfying $\mathrm{Ext}^1_\Lambda(M, Y) \neq 0$.

(b) Suppose $\mathcal{X}$ is covariantly finite. We put $\sigma M = 0$ if M is Ext-injective. If M is not Ext-injective, then σM is the unique indecomposable direct summand Z of $X^{\mathrm{TrD}\, M}$ satisfying $\mathrm{Ext}^1_\Lambda(Z, M) \neq 0$.

COROLLARY 2.4. *Let* $0 \to A \to B \to C \to 0$ *be an almost split sequence in* $\mathcal{X}$.

(a) *If* $\mathcal{X}$ *is contravariantly finite*, $A \cong \tau C$.

(b) *If* $\mathcal{X}$ *is covariantly finite*, $C \cong \sigma A$.

Proof. (a) C must not be Ext-projective in $\mathcal{X}$, so that we can proceed as in the proof of Theorem 2.3(a) and get the commutative diagram (2.1) in which the middle row does not split. It follows that h does not factor through f, hence, must be a split monomorphism. Thus A is an indecomposable direct summand of $X_{\mathrm{DTr}\, C}$ satisfying $\mathrm{Ext}^1_\Lambda(C, A) \neq 0$. Using Notation 2.1(a) and Theorem 2.3(a), we get $A \cong \tau C$.

(b) This is dual to the proof of (a). ∎

We now obtain two one-sided versions of Theorem 2.4 of [AS81a], which is the main existence theorem of that paper. Combining the two one-sided versions, we get a new proof of the theorem, one that does not rely on the theory of dualizing R-varieties.

First, we recall the notions of covariant and contravariant defect of a short exact sequence [ARS Sect. IV.4, p. 128]. Given an exact sequence $\delta : 0 \to L \to M \to N \to 0$ in $\operatorname{mod} \Lambda$, the covariant defect δ_* and contravariant defect δ^* of δ are the subfunctors of $\mathrm{Ext}^1_\Lambda(N,\ )$ and $\mathrm{Ext}^1_\Lambda(\ , L)$, respectively, defined by the exact sequences

$$0 \to \mathrm{Hom}_\Lambda(N,\ ) \to \mathrm{Hom}_\Lambda(M,\ ) \to \mathrm{Hom}_\Lambda(L,\ ) \to \delta_* \to 0$$

and

$$0 \to \mathrm{Hom}_\Lambda(\ , L) \to \mathrm{Hom}_\Lambda(\ , M) \to \mathrm{Hom}_\Lambda(\ , N) \to \delta^* \to 0$$

in $\operatorname{mod} R$.

For all finitely generated R-modules U, denote by $\langle U \rangle$ the R-length of U.

PROPOSITION 2.5. *Let* $\delta : 0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ *be an exact sequence in* X. *For all* $Z \in \operatorname{mod} \Lambda$:

(a) *The morphism* $\mathrm{Hom}_\Lambda(f^Z, N) : \mathrm{Hom}_\Lambda(X^Z, N) \to \mathrm{Hom}_\Lambda(Z, N)$ *of* R*-modules induces an isomorphism*

$$\mathrm{Hom}_\Lambda(X^Z, N)/\mathrm{Im}\, \mathrm{Hom}_\Lambda(X^Z, g) \xrightarrow{\sim} \mathrm{Hom}_\Lambda(Z, N)/\mathrm{Im}\, \mathrm{Hom}_\Lambda(Z, g).$$

In particular, $\langle \delta^*(Z) \rangle = \langle \delta^*(X^Z) \rangle$.

(b) *The morphism* $\mathrm{Hom}_\Lambda(L, g_Z) : \mathrm{Hom}_\Lambda(L, X_Z) \to \mathrm{Hom}_\Lambda(L, Z)$ *of* R*-modules induces an isomorphism*

$$\mathrm{Hom}_\Lambda(L, X_Z)/\mathrm{Im}\, \mathrm{Hom}_\Lambda(f, X_Z) \xrightarrow{\sim} \mathrm{Hom}_\Lambda(L, Z)/\mathrm{Hom}_\Lambda(f, Z).$$

In particular, $\langle \delta_*(X_Z) \rangle = \langle \delta_*(Z) \rangle$.

Proof. (a) Since $N \in \mathsf{X}$ and f^Z is a left X-approximation of Z, then $\mathrm{Hom}_\Lambda(f^Z, N)$ is an epimorphism of R-modules, so that it suffices to show that $\mathrm{Im}\, \mathrm{Hom}_\Lambda(X^Z, g)$ is the full preimage of $\mathrm{Im}\, \mathrm{Hom}_\Lambda(Z, g)$ under $\mathrm{Hom}_\Lambda(f^Z, N)$. It is clear that

$$\mathrm{Hom}_\Lambda(f^Z, N)\big(\mathrm{Im}\, \mathrm{Hom}_\Lambda(X^Z, g)\big) \subset \mathrm{Im}\, \mathrm{Hom}_\Lambda(Z, g).$$

We have to show that if $h: X^Z \to N$ has the property that hf^Z factors through g, then h factors through g.

Consider the exact commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r] \arrow[d, equal] & U \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d, "f^Z"] & 0 \\
0 \arrow[r] & L \arrow[r] \arrow[d, equal] & V \arrow[r] \arrow[d] & X^Z \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & L \arrow[r, "f"] & M \arrow[r, "g"] & N \arrow[r] & 0
\end{tikzcd}$$

in mod Λ. Since hf^Z factors through g, the top row splits.

Let $\alpha \in \operatorname{Ext}^1_\Lambda(Z, L)$ and $\beta \in \operatorname{Ext}^1_\Lambda(X^Z, L)$ be the elements corresponding to the top and middle rows of the diagram, respectively. Then $0 = \alpha = \operatorname{Ext}^1_\Lambda(f^Z, L)(\beta)$. Since $\operatorname{Ext}^1_\Lambda(f^Z, L)$ is a monomorphism of R-modules by Lemma 2.1(a), we have $\beta = 0$, i.e., the middle row splits. It follows that h factors through g.

(b) This is dual to the proof of (a). ∎

THEOREM 2.6. (a) *Assume* X *is contravariantly finite. If* $C \in \mathsf{X}$ *is an indecomposable non-Ext-projective module, then there exists an exact sequence* $0 \to X_{\mathrm{DTr}\, C} \xrightarrow{g} E \xrightarrow{f} C \to 0$ *with* f *a right almost split morphism in* X.

(b) *Assume* X *is covariantly finite. If* $A \in \mathsf{X}$ *is an indecomposable non-Ext-injective module, then there exists an exact sequence* $0 \to A \xrightarrow{g} F \xrightarrow{f} X^{\mathrm{TrD}\, A} \to 0$ *with* g *a left almost split morphism in* X.

Proof. (a) Since C is not Ext-projective, there exists a non-split exact sequence $\delta: 0 \to L \to M \to C \to 0$ in X. Since not every endomorphism of C can be lifted to M, we have $\langle \delta^*(C) \rangle \neq 0$. By IV Theorem 4.1 of [ARS], $\langle \delta^*(C) \rangle = \langle \delta_*(\mathrm{DTr}\, C) \rangle$, and Proposition 2.5(b) gives $\langle \delta_*(X_{\mathrm{DTr}\, C}) \rangle = \langle \delta_*(\mathrm{DTr}\, C) \rangle$, so that $\langle \delta_*(X_{\mathrm{DTr}\, C}) \rangle \neq 0$. It follows that $X_{\mathrm{DTr}\, C}$ is not Ext-injective, whence there exists a non-split exact sequence $0 \to X_{\mathrm{DTr}\, C} \xrightarrow{h} B \xrightarrow{j} V \to 0$ in X. Since not every endomorphism of $X_{\mathrm{DTr}\, C}$ factors through h, Proposition 2.5(b) and IV Theorem 4.1 of [ARS] imply that not every morphism $C \to V$ factors through j.

From now on, the proof proceeds as that of V Theorem 1.15(a) of [ARS], starting with line 14 from the top of p. 145 to the end on p. 146. One only has to replace everywhere $\mathrm{DTr}\, C$ by $X_{\mathrm{DTr}\, C}$ and replace the reference to IV Corollary 4.4 of [ARS] in the top line of p. 146 by the reference to Proposition 2.5(b) of the present paper and IV Theorem 4.1 of [ARS], keeping in mind that $\mathcal{X}$ is closed under extensions.

(b) This is dual to the proof of (a). ∎

The following statement is a consequence of Proposition 4.4 from Chapter II of [A], which was proved with the use of a duality between certain categories of finitely presented functors. For the convenience of the reader, we give an elementary proof.

PROPOSITION 2.7. (a) *Let* $0 \to \operatorname{Ker} q \xrightarrow{p} B \xrightarrow{q} C \to 0$ *be an exact sequence in* $\mathcal{X}$ *with* q *a minimal right almost split morphism in* $\mathcal{X}$. *Then* $\operatorname{Ker} q$ *is indecomposable and* p *is a minimal left almost split morphism in* $\mathcal{X}$.

(b) *Let* $0 \to A \xrightarrow{p} E \xrightarrow{q} \operatorname{Coker} p \to 0$ *be an exact sequence in* $\mathcal{X}$ *with* p *a minimal left almost split morphism in* $\mathcal{X}$. *Then* $\operatorname{Coker} p$ *is indecomposable and* q *is a minimal right almost split morphism in* $\mathcal{X}$.

Proof. (a) According to Theorem 2.3(b), the exact sequence $0 \to X_{\mathrm{DTr}\, C} \xrightarrow{g} E \xrightarrow{f} C \to 0$ given by Theorem 2.6(a) is the direct sum of a split exact sequence $0 \to Y \to Y \to 0 \to 0$ and a non-split exact sequence $0 \to \tau C \to U \xrightarrow{l} C \to 0$, where l is a minimal right almost split morphism in $\mathcal{X}$ because τC is indecomposable. By the uniqueness of a minimal right almost split morphism in $\mathcal{X}$, we have $\operatorname{Ker} q \cong \tau C$, whence $\operatorname{Ker} q$ is indecomposable. p is a left minimal morphism because C is indecomposable.

To prove that p is a left almost split morphism in $\mathcal{X}$, it suffices to show that if $X \in \mathcal{X}$ is indecomposable and a morphism $\alpha : \operatorname{Ker} q \to X$ does not factor through p, then α is an isomorphism. Consider the pushout diagram

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Ker} q \arrow[r, "p"] \arrow[d, "\alpha"'] & B \arrow[r, "q"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & X \arrow[r, "t"] & F \arrow[r, "s"] & C \arrow[r] & 0
\end{tikzcd}$$

and note that $F \in \mathcal{X}$ because $\mathcal{X}$ is closed under extensions. Since α does not factor through p, the bottom row does not split, whence $s = q\delta$ for some $\delta : F \to B$ because q is right almost split in $\mathcal{X}$. Denote by $\gamma : X \to \operatorname{Ker} q$ the unique morphism in $\operatorname{mod} \Lambda$ satisfying $\delta t = p\gamma$; we obtain the exact commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Ker} q \arrow[r, "p"] \arrow[d, "\alpha"'] & B \arrow[r, "q"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & X \arrow[r, "t"] \arrow[d, "\gamma"'] & F \arrow[r, "s"] \arrow[d, "\delta"] & C \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \operatorname{Ker} q \arrow[r, "p"] & B \arrow[r, "q"] & C \arrow[r] & 0.
\end{tikzcd}$$

From the equality $q = s\beta = q\,\delta\beta$, we conclude that $\delta\beta$ is an isomorphism because q is a right minimal morphism. Then $\gamma\alpha$ is an isomorphism, so that α is an isomorphism because X is indecomposable.

(b) This is dual to the proof of (a). ∎

COROLLARY 2.8. *Assume X is contravariantly finite and let $C \in \mathsf{X}$ be an indecomposable module.*

(a) *There exists a right almost split morphism $B \to C$ in X.*

(b) *If C is not Ext-projective, there exists an almost split sequence $0 \to \tau C \to E \to C \to 0$ in X.*

Proof. (a) See Proposition 3.10 of [AS80].

(b) This follows from Theorem 2.6(a), Proposition 2.7(a), and Corollary 2.4(a). ∎

For the sake of completeness, we state without proof the result dual to Corollary 2.8.

COROLLARY 2.9. *Assume X is covariantly finite and let $A \in \mathsf{X}$ be an indecomposable module.*

(a) *There exists a left almost split morphism $A \to B$ in X.*

(b) *If A is not Ext-injective, there exists an almost split sequence $0 \to A \to E \to \sigma A \to 0$ in X.*

It follows from Corollaries 2.8, 2.9, and Theorem 2.3 that if X is covariantly (contravariantly) finite and A (C) is not Ext-injective (Ext-projective), then σA (τC) is a uniquely determined direct summand of $X^{\mathrm{TrD}\,A}$ ($X_{\mathrm{DTr}\,C}$). It would be interesting to describe all covariantly (contravariantly) finite subcategories X of mod Λ with the property that for all A (C), module $X^{\mathrm{TrD}\,A}$ ($X_{\mathrm{DTr}\,C}$) is indecomposable and, consequently, $\sigma A \cong X^{\mathrm{TrD}\,A}$ ($\tau C \cong X_{\mathrm{DTr}\,C}$). We will give examples of such subcategories in Section 4.

We finish the section with the following uniqueness result. For $M \in$ mod Λ denote by $\underline{\mathrm{End}}_\Lambda(M)$ ($\overline{\mathrm{End}}_\Lambda(M)$) the factor ring of $\mathrm{End}_\Lambda(M)$ modulo the ideal of all morphisms $M \to M$ that factor through a projective (injective) module.

PROPOSITION 2.10. (a) *Assume $\mathcal{X}$ is contravariantly finite. Let* $0 \to X_{\mathrm{DTr}\,C} \xrightarrow{s} U \xrightarrow{t} C \to 0$ *be a non-split exact sequence in $\mathcal{X}$, where C is indecomposable with* $\underline{\mathrm{End}}_\Lambda(C)$ *a division ring. Then the bottom row of the pushout diagram*

$$\begin{tikzcd} 0 \arrow[r] & X_{\mathrm{DTr}\,C} \arrow[r, "s"] \arrow[d, "g_{\mathrm{DTr}\,C}"'] & U \arrow[r, "t"] \arrow[d] & C \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathrm{DTr}\,C \arrow[r] & V \arrow[r] & C \arrow[r] & 0 \end{tikzcd}$$

is an almost split sequence in $\mathrm{mod}\,\Lambda$, *and t is a right almost split morphism in* $\mathcal{X}$.

(b) *Assume $\mathcal{X}$ is covariantly finite. Let* $0 \to A \xrightarrow{p} W \xrightarrow{q} X^{\mathrm{TrD}\,A} \to 0$ *be a non-split exact sequence in $\mathcal{X}$, where A is indecomposable with* $\overline{\mathrm{End}}_\Lambda(A)$ *a division ring. Then the top row of the pullback diagram*

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, equal] & Y \arrow[r] \arrow[d] & \mathrm{TrD}\,A \arrow[r] \arrow[d, "f^{\mathrm{TrD}\,A}"] & 0 \\ 0 \arrow[r] & A \arrow[r, "p"] & W \arrow[r, "q"] & X^{\mathrm{TrD}\,A} \arrow[r] & 0 \end{tikzcd}$$

is an almost split sequence in $\mathrm{mod}\,\Lambda$, *and p is a left almost split morphism in* $\mathcal{X}$.

Proof. (a) By V Corollary 2.4(a) of [ARS], if the bottom row does not split, then it is an almost split sequence in $\mathrm{mod}\,\Lambda$. Suppose, to the contrary, that it does split. Then $g_{\mathrm{DTr}\,C} = hs$ for some $h : U \to \mathrm{DTr}\,C$. Since $U \in \mathcal{X}$, we have $h = g_{\mathrm{DTr}\,C}\,j$ for some $j : U \to X_{\mathrm{DTr}\,C}$. Therefore, $g_{\mathrm{DTr}\,C} = g_{\mathrm{DTr}\,C}\,js$, whence js is an isomorphism because $g_{\mathrm{DTr}\,C}$ is a right minimal morphism. It follows that the top row splits, a contradiction.

By Theorem 2.3(b), the exact sequence $0 \to X_{\mathrm{DTr}\,C} \xrightarrow{s} U \xrightarrow{t} C \to 0$ is the direct sum of a split exact sequence $0 \to Y \to Y \to 0 \to 0$ and a non-split exact sequence $0 \to \tau C \to E \to C \to 0$. In view of Corollary 2.8(b), an argument similar to the proof of V Proposition 2.3 of [ARS] shows that the latter sequence must be almost split in $\mathcal{X}$.

(b) This is dual to the proof (a). ∎

3. EXT-PROJECTIVE AND EXT-INJECTIVE MODULES

In this section, we study right (left) almost split morphisms in $\mathcal{X}$ with Ext-projective codomain (Ext-injective domain); obtain a new characterization of Ext-projective (Ext-injective) modules in terms of minimal right (left) $\mathcal{X}$-approximations, DTr (TrD), and functor Ext^1_Λ; and describe the subcategory of mod Λ determined by the Ext-projective (Ext-injective) modules in terms of left (right) $\mathcal{X}$-approximations.

LEMMA 3.1. *Let $X \in \mathcal{X}$ be an indecomposable module.*

(a) *If X is Ext-projective but not splitting projective, then for any non-split short exact sequence $0 \to \operatorname{Ker} g \xrightarrow{f} Y \xrightarrow{g} X \to 0$ with $Y \in \mathcal{X}$, the morphism f is a minimal left $\mathcal{X}$-approximation of* $\operatorname{Ker} g$. *If g is a right minimal morphism,* $\operatorname{Ker} g$ *is indecomposable.*

(b) *If $\mathcal{X}$ is Ext-injective but not splitting injective, then for any non-split short exact sequence $0 \to X \xrightarrow{f} Y \xrightarrow{g} \operatorname{Coker} f \to 0$ with $Y \in \mathcal{X}$, the morphism g is a minimal right $\mathcal{X}$-approximation of* $\operatorname{Coker} f$. *If f is a left minimal morphism,* $\operatorname{Coker} f$ *is indecomposable.*

Proof. (a) Given a morphism $h : \operatorname{Ker} g \to A$ with $A \in \mathcal{X}$, consider the pushout diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{Ker} g & \xrightarrow{f} & Y & \xrightarrow{g} & X & \longrightarrow & 0 \\
 & & {\scriptstyle h}\downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & X & \longrightarrow & 0.
\end{array}$$

Since X is Ext-projective, the bottom row splits, so that h factors through f. Since the top row is a non-split exact sequence and X is indecomposable, f is a left minimal morphism. Thus, f is a minimal left $\mathcal{X}$-approximation of $\operatorname{Ker} g$.

Suppose now that g is a right minimal morphism and show $\operatorname{Ker} g$ is indecomposable. Since a finite direct sum of left minimal morphisms is a left minimal morphism, f is the direct sum of minimal left $\mathcal{X}$-approximations $f^Z : Z \to X^Z$ of the indecomposable direct summands Z of $\operatorname{Ker} g$. Then each f^Z is a monomorphism and $X \cong \operatorname{Coker} f \cong \bigoplus_Z \operatorname{Coker} f^Z$, where $\operatorname{Coker} f^Z \neq 0$ for all Z because g is right minimal. Since X is indecomposable, so is $\operatorname{Ker} g$.

(b) This is dual to the proof of (a). ∎

PROPOSITION 3.2. *Assume $\mathcal{X}$ is contravariantly finite. Let $g : B \to C$ be a minimal right almost split morphism in $\mathcal{X}$ with C an Ext-projective module.*

(a) *g is not surjective if and only if C is splitting projective.*

(b) *Suppose C is not splitting projective. The short exact sequence $0 \to \operatorname{Ker} g \xrightarrow{f} B \xrightarrow{g} C \to 0$ has the property that* $\operatorname{Ker} g$ *is an indecomposable module not in $\mathcal{X}$ and f is a minimal left $\mathcal{X}$-approximation of* $\operatorname{Ker} g$. *If*

$g' : B' \to C'$ is a minimal right almost split morphism in $\mathcal{X}$ with C' Ext-projective but not splitting projective, then $C' \cong C$ if and only if $\operatorname{Ker} g' \cong \operatorname{Ker} g$.

Proof. (a) For the necessity, note that since g is not surjective, no surjection $X \to C$ with $X \in \mathcal{X}$ factors through g. Hence every such surjection is a split epimorphism, i.e., C is splitting projective. The sufficiency follows from the fact that g is not a split epimorphism.

(b) By (a), g is surjective, so that the indicated sequence is exact. Since g is not a split epimorphism and C is Ext-projective, $\operatorname{Ker} g \notin \mathcal{X}$. The rest of the first assertion follows from Lemma 3.1(a), taking into account that C is indecomposable.

For the second assertion, consider the short exact sequence $0 \to \operatorname{Ker} g' \xrightarrow{f'} B' \xrightarrow{g'} C' \to 0$. If $h : C' \to C$ is an isomorphism, there is an isomorphism $j : B' \to B$ satisfying $hg' = gj$, because a minimal right almost split morphism in $\mathcal{X}$ with codomain C is unique up to isomorphism. Hence $\operatorname{Ker} g' \cong \operatorname{Ker} g$. If $l : \operatorname{Ker} g' \cong \operatorname{Ker} g$ is an isomorphism, there exists an isomorphism $j : B' \to B$ satisfying $jf' = fl$, because a minimal left $\mathcal{X}$-approximation of $\operatorname{Ker} g'$ is unique up to isomorphism. Hence $C' \cong C$. ∎

For the convenience of the reader, we give without proof the statement dual to Proposition 3.2.

PROPOSITION 3.3. *Assume $\mathcal{X}$ is covariantly finite. Let $f : A \to B$ be a minimal left almost split morphism in $\mathcal{X}$ with A an Ext-injective module.*

(a) *f is not injective if and only if A is splitting injective.*

(b) *Suppose A is not splitting injective. The short exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} \operatorname{Coker} f \to 0$ has the property that $\operatorname{Coker} f$ is an indecomposable module not in $\mathcal{X}$ and g is a minimal right $\mathcal{X}$-approximation of $\operatorname{Coker} f$. If $f' : A' \to B'$ is a minimal left almost split morphism in $\mathcal{X}$ with A' Ext-injective but not splitting injective, then $A' \cong A$ if and only if $\operatorname{Coker} f' \cong \operatorname{Coker} f$.*

Remark 3.1. The short exact sequence in the statement of Proposition 3.2(b) is not an exact sequence in $\mathcal{X}$ because $\operatorname{Ker} g \notin \mathcal{X}$. However, it behaves very much like an almost split sequence in $\mathcal{X}$: g is a minimal right almost split morphism in $\mathcal{X}$; the left end-term is indecomposable and uniquely determined by C; for all $X \in \mathcal{X}$, every morphism $\operatorname{Ker} g \to X$ that is not a split monomorphism (i.e., every morphism $\operatorname{Ker} g \to X$, because $\operatorname{Ker} g \notin \mathcal{X}$) factors through f.

It would be interesting to have a formula for $\operatorname{Ker} g$ in terms of C.

The dual comments apply to the short exact sequence in the statement of Proposition 3.3(b).

Our next result is a characterization of Ext-projective and Ext-injective modules in a contravariantly finite subcategory of mod Λ.

THEOREM 3.4. *Assume $\mathcal{X}$ is contravariantly finite and let $M \in \mathcal{X}$ be an indecomposable module.*

(a) *M is an Ext-projective module if and only if* $\mathrm{Ext}^1_\Lambda(M, X_{\mathrm{DTr}\,M}) = 0$.

(b) *M is a splitting injective module if and only if it is a direct summand of X_I for some indecomposable injective Λ-module I.*

(c) *M is an Ext-injective module if and only if it is either a splitting injective module or a direct summand of $X_{\mathrm{Ker}\,g_Z}$ for some $Z \in$* mod Λ.

Proof. (a) The necessity follows from the definition of an Ext-projective module. For the sufficiency, note that if M is not Ext-projective, Theorem 2.3(a) implies $\mathrm{Ext}^1_\Lambda(M, X_{\mathrm{DTr}\,M}) \neq 0$ because functor $\mathrm{Ext}^1_\Lambda(M, \ )$ commutes with finite direct sums. Since $\mathrm{Ext}^1_\Lambda(M, X_{\mathrm{DTr}\,M}) = 0$ by assumption, M must be Ext-projective.

(b) See Lemma 3.11(a) and Proposition 3.6, both of [AS80].

(c) For the necessity, suppose M is an Ext-injective but not splitting injective module and show it is a direct summand of $X_{\mathrm{Ker}\,g_Z}$ for some $Z \in \mathrm{mod}\,\Lambda$. According to Lemma 3.11(a) and Corollary 2.4(d), both of [AS80], there exists a non-split short exact sequence $0 \to M \xrightarrow{f} N \xrightarrow{g} \mathrm{Coker}\,f \to 0$ with N a splitting injective module in $\mathcal{X}$. By Lemma 3.1(b), g is a minimal right $\mathcal{X}$-approximation of $\mathrm{Coker}\,f$, whence $g = g_{\mathrm{Coker}\,f}$ and $M = \mathrm{Ker}\,g_{\mathrm{Coker}\,f} = X_{\mathrm{Ker}(g_{\mathrm{Coker}\,f})}$.

For the sufficiency, note that if M is splitting injective, then, obviously, M is Ext-injective. Show that if M is a direct summand of $X_{\mathrm{Ker}\,g_Z}$ for some $Z \in \mathrm{mod}\,\Lambda$, then M is an Ext-injective module. We have the exact sequence

$$0 \to \mathrm{Ker}\,g_Z \to X_Z \xrightarrow{g_Z} Z$$

and a minimal right $\mathcal{X}$-approximation $g_{\mathrm{Ker}\,g_Z} : X_{\mathrm{Ker}\,g_Z} \to \mathrm{Ker}\,g_Z$. By Wakamatsu's Lemma, $\mathrm{Ext}^1_\Lambda(\ , \mathrm{Ker}\,g_Z) \,|\, \mathcal{X} = 0$. By Lemma 2.1(b),

$$\mathrm{Ext}^1_\Lambda(\ , g_{\mathrm{Ker}\,g_Z}) : \mathrm{Ext}^1_\Lambda(\ , X_{\mathrm{Ker}\,g_Z}) \,|\, \mathcal{X} \to \mathrm{Ext}^1_\Lambda(\ , \mathrm{Ker}\,g_Z) \,|\, \mathcal{X}$$

is a monomorphism, whence $\mathrm{Ext}^1_\Lambda(\ , X_{\mathrm{Ker}\,g_Z}) \,|\, \mathcal{X} = 0$, i.e., $X_{\mathrm{Ker}\,g_Z}$ is Ext-injective. Since M is a direct summand of $X_{\mathrm{Ker}\,g_Z}$ by assumption, M is Ext-injective. ∎

The utility of part (a) of Theorem 3.4 is that in order to verify whether the given module M is Ext-projective, i.e., whether $\mathrm{Ext}^1_\Lambda(M, \mathcal{X}) = 0$ for all

$X \in \mathcal{X}$, it suffices to verify whether $\operatorname{Ext}^1_\Lambda(M, X) = 0$ for just one X, namely, for the module $X = X_{\mathrm{DTr}\, M}$, which can be computed from M.

We now introduce the following definitions and notation. For a subcategory $\mathcal{Y}$ of mod Λ, denote by Ind $\mathcal{Y}$ a complete set of pairwise non-isomorphic indecomposable direct summands of modules in $\mathcal{Y}$. Let $\mathcal{C}$ be a subcategory of mod Λ. If $\mathcal{X}$ is contravariantly finite, we call the set

$$\mathcal{X}_{\mathcal{C}} = \{M \in \operatorname{Ind} \mathcal{X} \,|\, M \text{ is a direct summand of } X_C \text{ for some } C \in \mathcal{C}\}$$

the minimal right $\mathcal{X}$-approximation of $\mathcal{C}$. If $\mathcal{X}$ is covariantly finite, we call the set

$$\mathcal{X}^{\mathcal{C}} = \{M \in \operatorname{Ind} \mathcal{X} \,|\, M \text{ is a direct summand of } X^C \text{ for some } C \in \mathcal{C}\}$$

the minimal left $\mathcal{X}$-approximation of $\mathcal{C}$. Let $\mathcal{K}(\mathcal{X})$ be the subcategory of mod Λ determined by all modules of the form $\operatorname{Ker} g_Z$ for some $Z \in \operatorname{mod} \Lambda$, and let $\mathcal{C}(\mathcal{X})$ be the subcategory determined by all modules of the form $\operatorname{Coker} f^Z$ for some $Z \in \operatorname{mod} \Lambda$. Denote by $\mathcal{I}_0(\mathcal{X})$ ($\mathcal{P}_0(\mathcal{X})$) a complete set of pairwise non-isomorphic indecomposable splitting injective (projective) modules in $\mathcal{X}$.

For a set S, denote by $|S|$ the cardinality of S.

COROLLARY 3.5. *Assume $\mathcal{X}$ is contravariantly finite. $\mathcal{X}_{\mathcal{K}(\mathcal{X})} \cup \mathcal{I}_0(\mathcal{X})$ is a complete set of non-isomorphic indecomposable Ext-injective modules in $\mathcal{X}$. $\mathcal{X}$ has finitely many non-isomorphic indecomposable Ext-injective modules if and only if $|\mathcal{X}_{\mathcal{K}(\mathcal{X})}| < \infty$.*

For the sake of completeness, we present without proof the results dual to Theorem 3.4 and Corollary 3.5.

THEOREM 3.6. *Assume $\mathcal{X}$ is covariantly finite and let $M \in \mathcal{X}$ be an indecomposable module.*

(a) *M is an Ext-injective module if and only if* $\operatorname{Ext}^1_\Lambda(X^{\mathrm{TrD}\, M}, M) = 0$.

(b) *M is a splitting projective module if and only if it is a direct summand of X^P for some indecomposable projective Λ-module P.*

(c) *M is an Ext-projective module if and only if it is either a splitting projective module or a direct summand of $X^{\operatorname{Coker} f^Z}$ for some $Z \in$* mod Λ.

COROLLARY 3.7. *Assume $\mathcal{X}$ is covariantly finite. $\mathcal{X}^{\mathcal{C}(\mathcal{X})} \cup \mathcal{P}_0(\mathcal{X})$ is a complete set of non-isomorphic indecomposable Ext-projective modules in $\mathcal{X}$. $\mathcal{X}$ has finitely many non-isomorphic indecomposable Ext-projective modules if and only if $|\mathcal{X}^{\mathcal{C}(\mathcal{X})}| < \infty$.*

Putting together Theorems 3.4, 3.6, and Corollaries 3.5, 3.7, we obtain the following.

COROLLARY 3.8. *Assume* $\mathcal{X}$ *is functorially finite and let* $M \in \mathcal{X}$ *be an indecomposable module.*

(a) *The following are equivalent*:

(i) *M is Ext-projective.*

(ii) $\operatorname{Ext}^1_\Lambda(M, X_{\mathrm{DTr}\,M}) = 0$.

(iii) *M is either a splitting projective module or a direct summand of* $X^{\operatorname{Coker} f^Z}$ *for some* $Z \in \operatorname{mod}\Lambda$.

(b) *The following are equivalent*:

(i) *M is Ext-injective.*

(ii) $\operatorname{Ext}^1_\Lambda(X^{\mathrm{TrD}\,M}, M) = 0$.

(iii) *M is either a splitting injective module or a direct summand of* $X_{\operatorname{Ker} g_Z}$ *for some* $Z \in \operatorname{mod}\Lambda$.

We finish this section by giving a sufficient condition for the finiteness of number of non-isomorphic indecomposable Ext-projective (Ext-injective) modules in a homologically finite subcategory associated with a pair of adjoint functors.

Let Σ (Θ) be an artin algebra. Denote by $\operatorname{mod}\Sigma$ $(\operatorname{mod}\Theta)$ the category of finitely generated left Σ-modules (Θ-modules) and let $F : \operatorname{mod}\Sigma \to \operatorname{mod}\Theta$ be a functor. Denote by $\operatorname{Im} F$ the subcategory of $\operatorname{mod}\Theta$ consisting of all modules isomorphic to FX for some $X \in \operatorname{mod}\Sigma$. For a subcategory $\mathcal{C}$ $(\mathcal{D})$ of $\operatorname{mod}\Sigma$ $(\operatorname{mod}\Theta)$, denote by $\operatorname{add}\mathcal{C}$ $(\operatorname{add}\mathcal{D})$ the subcategory consisting of all Σ (Θ)-modules isomorphic to direct summands of finite direct sums of modules in $\mathcal{C}$ $(\mathcal{D})$.

PROPOSITION 3.9. *Let* Σ *and* Θ *be artin algebras and let* $S : \operatorname{mod}\Sigma \to \operatorname{mod}\Theta$, $T : \operatorname{mod}\Theta \to \operatorname{mod}\Sigma$ *be a pair of additive adjoint functors with S a left adjoint of T.*

(a) *Assume* $\operatorname{add}(\operatorname{Im} S)$ *is closed under extensions. If*

$$\left|\operatorname{Ind} T\mathcal{K}(\operatorname{add}(\operatorname{Im} S))\right| < \infty,$$

then $\operatorname{add}(\operatorname{Im} S)$ *has finitely many non-isomorphic indecomposable Ext-injective modules.*

(b) *Assume* $\operatorname{add}(\operatorname{Im} T)$ *is closed under extensions. If*

$$\left|\operatorname{Ind} S\mathcal{C}(\operatorname{add}(\operatorname{Im} T))\right| < \infty,$$

then $\operatorname{add}(\operatorname{Im} T)$ *has finitely many non-isomorphic indecomposable Ext-projective modules.*

Proof. (a) By Proposition 1.2(a) of [AR92], $\operatorname{add}(\operatorname{Im} S)$ is contravariantly finite and each $N \in \operatorname{mod}\Theta$ has a right $\operatorname{add}(\operatorname{Im} S)$-approximation $STN \to$

N. If $\operatorname{Ind} T\mathcal{K}(\operatorname{add}(\operatorname{Im} S)) = \{L_1, \dots, L_t\}$, then for $N \in \mathcal{K}(\operatorname{add}(\operatorname{Im} S))$, we have a right $\operatorname{add}(\operatorname{Im} S)$-approximation $\oplus_{i=1}^{t}(SL_i)^{m_i} \to N$ because S is an additive functor. It follows that

$$\left|\operatorname{Ind} \operatorname{add}(\operatorname{Im} S)_{\mathcal{K}(\operatorname{add}(\operatorname{Im} S))}\right| < \infty.$$

By Corollary 3.5, $\operatorname{add}(\operatorname{Im} S)$ has finitely many non-isomorphic indecomposable Ext-injective modules.

(b) This is dual to the proof of (a). ∎

4. SUBCATEGORIES CLOSED UNDER SUBMODULES

In this section, we assume that, in addition to being closed under extensions and satisfying other conditions indicated in the Introduction, $\mathcal{X}$ is a subcategory of $\operatorname{mod} \Lambda$ closed under submodules. We apply the general theory of Sections 2 and 3 to obtain, refine, or generalize some of the results of [AS81a, BM] on such an $\mathcal{X}$. The dual consideration of subcategories closed under factor modules can be carried out similarly, using the results of Sections 3 and 4 of [AS81a]; we leave this to the reader.

According to Proposition 4.8(b) of [AS80], for all $Z \in \operatorname{mod} \Lambda$, there exists a unique minimal submodule $t_{\mathcal{X}}(Z)$ among the submodules Y of Z satisfying $Z/Y \in \mathcal{X}$. It follows that $\mathcal{X}$ is a covariantly finite subcategory of $\operatorname{mod} \Lambda$ with the natural projection $f^Z : Z \to Z/t_{\mathcal{X}}(Z)$ being a minimal left $\mathcal{X}$-approximation of Z; in particular, $X^Z = Z/t_{\mathcal{X}}(Z)$. By Proposition 4.7(c) of [AS80], $\mathcal{X}$ is functorially finite if and only if $\mathcal{X} = \operatorname{Sub} M$ for some $M \in \operatorname{mod} \Lambda$.

Parts (a), (b), and (d) of the following statement are Corollaries 3.4, 3.5, and Proposition 3.1(c) of [AS81a]. Parts (b) and (c) were obtained in [BM]; part (c) was obtained in [H]. We show how to obtain (a), (b), and (d) as consequences of our general theory. Although we have nothing new to say about the proof of (c), we quote the result because it provides an example of a large class of covariantly finite subcategories $\mathcal{X}$ for which the minimal left approximation of $\operatorname{TrD} A$ is indecomposable for all indecomposable non-Ext-injective modules $A \in \mathcal{X}$.

PROPOSITION 4.1. *Suppose $A \in \mathcal{X}$ is an indecomposable module.*

(a) *A is Ext-injective if and only if* $t_{\mathcal{X}}(\operatorname{TrD} A) = \operatorname{TrD} A$.

(b) *If A is not Ext-injective, there exists an almost split sequence $0 \to A \to E \to \sigma A \to 0$ in $\mathcal{X}$, where σA is a direct summand of* $\operatorname{TrD} A/t_{\mathcal{X}}(\operatorname{TrD} A)$.

(c) *If A is not Ext-injective,* $\operatorname{TrD} A/t_{\mathcal{X}}(\operatorname{TrD} A)$ *is indecomposable.*

(d) *Every Ext-projective module in $\mathcal{X}$ is splitting projective.*

Proof. (a) By Theorem 3.6(a), A is Ext-injective if and only if $\mathrm{Ext}^1_\Lambda(X^{\mathrm{TrD}\,A}, A) = 0$. Since $X^{\mathrm{TrD}\,A} = \mathrm{TrD}\,A/t_{\mathsf{X}}(\mathrm{TrD}\,A)$, the sufficiency is obvious. Prove the necessity; we may assume A is not injective in $\mathrm{mod}\,\Lambda$.

Applying the contravariant functor $\mathrm{Hom}_\Lambda(\ ,A)$ to the short exact sequence $0 \to t_{\mathsf{X}}(\mathrm{TrD}\,A) \xrightarrow{g} \mathrm{TrD}\,A \xrightarrow{f^{\mathrm{TrD}\,A}} X^{\mathrm{TrD}\,A} \to 0$, we obtain the long exact sequence

$$\cdots \to \mathrm{Ext}^1_\Lambda(X^{\mathrm{TrD}\,A}, A) \to \mathrm{Ext}^1_\Lambda(\mathrm{TrD}\,A, A)$$
$$\xrightarrow[\mathrm{Ext}^1_\Lambda(g,\,A)]{} \mathrm{Ext}^1_\Lambda\big((t_{\mathsf{X}}(\mathrm{TrD}\,A), A)\big) \to \cdots.$$

Since A is Ext-injective, $\mathrm{Ext}^1_\Lambda(X^{\mathrm{TrD}\,A}, A) = 0$, so that $\mathrm{Ext}^1_\Lambda(g, A)$ is a monomorphism.

Consider the pullback diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, equal] & U \arrow[r] \arrow[d] & t_{\mathsf{X}}(\mathrm{TrD}\,A) \arrow[r] \arrow[d, "g"] & 0 \\
0 \arrow[r] & A \arrow[r, "s"] & E \arrow[r, "t"] & \mathrm{TrD}\,A \arrow[r] & 0,
\end{tikzcd}$$

where the bottom row is an almost split sequence in $\mathrm{mod}\,\Lambda$. Since $\mathrm{Ext}^1_\Lambda(g, A)$ is a monomorphism, the top row does not split, whence g does not factor through t. It follows that g is a split epimorphism, which must be an isomorphism. Thus $t_{\mathsf{X}}(\mathrm{TrD}\,A) = \mathrm{TrD}\,A$.

(b) This follows from Corollary 2.9(b) and Notation 2.1(b).

(c) See [BM, H].

(d) Follows from Theorem 3.6(c) in view of the fact that f^Z is an epimorphism for all $Z \in \mathrm{mod}\,\Lambda$. ▮

The next statement is Theorem 4.1 of [AS81a], which gives a description of the Ext-injective modules in $\mathsf{X} = \mathrm{Sub}\,M$. We apply a result from Section 3 in order to give a shorter proof of part (c) of that theorem. Our proof does not use categories of functors but relies on elementary properties of short exact sequences.

THEOREM 4.2. *Assume $\mathsf{X} = \mathrm{Sub}\,M$ for some $M \in \mathrm{mod}\,\Lambda$ and let $A \in \mathsf{X}$ be an indecomposable module.*

(a) *A is splitting injective if and only if it is a direct summand of X_I for some indecomposable injective Λ-module I.*

(b) *A is Ext-injective but not splitting injective if and only if A is a direct summand of $\mathrm{Ker}\,g_I$ for some indecomposable injective Λ-module I.*

Proof. We treat only the necessity of (b). Since X is contravariantly finite and A is not splitting injective, there exists a non-split exact sequence $0 \to A \overset{h}{\to} B \overset{j}{\to} C \to 0$ in $\operatorname{mod} \Lambda$ with B a splitting injective module in X. Let $\gamma : C \to I$ be a monomorphism with I injective in $\operatorname{mod} \Lambda$ and consider the exact sequence $0 \to \operatorname{Ker} g_I \overset{f}{\to} X_I \overset{g_I}{\to} I$. Since $B \in \mathsf{X}$, there exist a morphism $\beta : B \to X_I$ satisfying $\gamma_j = g_I \beta$ and a unique morphism $\alpha : A \to \operatorname{Ker} g_I$ satisfying $\beta h = f\alpha$. Since j is surjective, $\operatorname{Im} \gamma = \operatorname{Im} \gamma j \subset \operatorname{Im} g_I$, and we get the exact commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "h"] \arrow[d, "\alpha"'] & B \arrow[r, "j"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & \operatorname{Ker} g_I \arrow[r, "f"] & X_I \arrow[r, "g_I"] & \operatorname{Im} g_I \arrow[r] & 0
\end{tikzcd} \tag{4.1}$$

in $\operatorname{mod} \Lambda$.

Show that α and β are monomorphisms. According to Lemma III.1.3 of [M], there exists an exact commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "h"] \arrow[d, "\alpha"'] & B \arrow[r, "j"] \arrow[d, "\delta"] & C \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \operatorname{Ker} g_I \arrow[r] \arrow[d, equal] & E \arrow[r, "t"] \arrow[d, "\epsilon"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & \operatorname{Ker} g_I \arrow[r, "f"] & X_I \arrow[r, "g_I"] & \operatorname{Im} g_I \arrow[r] & 0
\end{tikzcd}$$

satisfying $\epsilon\delta = \beta$. Since γ is a monomorphism, so is ϵ, whence $E \in \mathsf{X}$ because X is closed under submodules. By Lemma 3.1(b), j is a minimal right X-approximation of C, so that $t = j\tau$ for some $\tau : E \to B$. We have $j = t\delta = j\tau\delta$, whence $\tau\delta$ is an isomorphism because j is a right minimal morphism. Hence δ is a monomorphism and so is $\beta = \epsilon\delta$. From the diagram (4.1), we conclude that α is a monomorphism.

Consider the commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A \arrow[r, "h"] \arrow[d, "\alpha"'] & B \arrow[r, "j"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & \operatorname{Ker} g_I \arrow[r, "f"] \arrow[d] & X_I \arrow[r, "g_I"] \arrow[d] & \operatorname{Im} g_I \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \operatorname{Coker} \alpha \arrow[r, "i"] \arrow[d] & \operatorname{Coker} \beta \arrow[r] \arrow[d] & \operatorname{Coker} \gamma \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

in which the columns and top two rows are exact. By the 3×3 lemma, the bottom row is exact.

Since B is a splitting injective module, β is a split monomorphism, whence $\operatorname{Coker} \beta \in \mathcal{X}$. Then i factors through a minimal left $\mathcal{X}$-approximation $f^{\operatorname{Coker} \alpha} : \operatorname{Coker} \alpha \to X^{\operatorname{Coker} \alpha}$, which is an epimorphism. Since i is a monomorphism, $f^{\operatorname{Coker} \alpha}$ must be an isomorphism, i.e., $\operatorname{Coker} \alpha \in \mathcal{X}$. Therefore the left column splits because A is Ext-injective, whence A direct summand of $\operatorname{Ker} g_I$. Since A is indecomposable, the desired result follows from the observation that a minimal right approximation of a direct sum is the direct sum of minimal right approximations. ∎

We finish this paper by presenting formulas for the left and right end-terms of an almost split sequence in $\mathcal{X} = \operatorname{Sub} \Lambda$. Recall that a maximal projective module is an indecomposable projective module P with the property that every monomorphism $P \to T$ with $T \in \operatorname{Sub} \Lambda$ splits, or, equivalently, every monomorphism $P \to Q$ with Q projective in $\operatorname{mod} \Lambda$ splits. For a module N, denote by $\operatorname{inj\,dim} N$ the injective dimension of N. We quote from the argument preceding Proposition 6.9 of [AS81a].

PROPOSITION 4.3. (a) *For an arbitrary artin algebra* Λ, *a complete set of non-isomorphic maximal projective modules forms a complete set of indecomposable splitting injective modules in* $\operatorname{Sub} \Lambda$.

(b) $\operatorname{Sub} \Lambda$ *is closed under extensions if and only if* $\operatorname{inj\,dim} P \leq 1$ *for all maximal projective* Λ-*modules* P.

Formulas for the left and right end-terms of an almost split sequence in $\operatorname{Sub} \Lambda$ were obtained in [BM] in the special case when Λ is a 1-Gorenstein artin algebra. Recall that Λ is 1-Gorenstein if an injective envelope of Λ is projective, which is equivalent to $\operatorname{inj\,dim} P = 0$ for all maximal projective modules P. In the situation when $\operatorname{inj\,dim}_{\Lambda} \Lambda = 1$, the formula for the left end-term of an almost split sequence in the category of Cohen−Macaulay modules over a Gorenstein artin algebra Λ [AR91b, Theorem 3.7, p. 234] is another special case, because in this situation the category of Cohen−Macaulay modules coincides with $\operatorname{Sub} \Lambda$. Recall that Λ is Gorenstein if $\operatorname{inj\,dim}_{\Lambda} \Lambda < \infty$ and $\operatorname{inj\,dim} \Lambda_{\Lambda} < \infty$. We consider the general case, in the sense that our only assumption is that $\mathcal{X} = \operatorname{Sub} \Lambda$ is closed under extensions.

Recall that since $\operatorname{Sub} M$, $M \in \operatorname{mod} \Lambda$, is a functorially finite subcategory, every module in $\operatorname{mod} \Lambda$ has a left (right) $\operatorname{Sub} M$-approximation. An explicit formula for a minimal left $\operatorname{Sub} M$-approximation of an arbitrary module in $\operatorname{mod} \Lambda$ was indicated above, but we have no explicit formula for a right $\operatorname{Sub} M$-approximation, whether minimal or not, in general. We can give such an explicit formula in case $M = \Lambda$; for the moment, we do not have to assume that $\operatorname{Sub} \Lambda$ is closed under extensions.

We remind the reader that for $N \in \operatorname{mod} \Lambda$, the first syzygy ΩN of N is defined by the exact sequence $0 \to \Omega N \to P_0(N) \xrightarrow{f_0(N)} N \to 0$, where $f_0(N)$ is a projective cover. The first cosyzygy $\Omega^{-1}N$ is defined by the exact sequence $0 \to N \xrightarrow{g_0(N)} I_0(N) \to \Omega^{-1}N \to 0$, where $g_0(N)$ is an injective envelope.

LEMMA 4.4. *Let $p: N \to I$ be an epimorphism, where $N \in \operatorname{Sub} \Lambda$ and $I \in \operatorname{mod} \Lambda$ is injective.*

(a) *p is a right* Sub Λ-*approximation of I.*

(b) *If p is a right minimal morphism, then N is a projective cover of I.*

Proof. (a) Let $h: X \to I$ be a morphism with $X \in \operatorname{Sub} \Lambda$, and let $i: X \to P$ be a monomorphism with P projective in $\operatorname{mod} \Lambda$. Denote by $j: P \to I$ a morphism satisfying $h = ji$, and by $q: P \to N$, a morphism satisfying $j = pq$. Then $h = pqi$, whence p is a right Sub Λ-approximation of I.

(b) Since a projective cover $P_0(I) \to I$ is an epimorphism and a right minimal morphism, (a) and the uniqueness of a minimal right Sub Λ-approximation imply $N \cong P_0(I)$. ∎

Lemma 4.4 says that a projective cover of an injective Λ-module is its minimal right Sub Λ-approximation. We now use the lemma to compute a right (not necessarily minimal) Sub Λ-approximation of an arbitrary module.

COROLLARY 4.5. *For $N \in \operatorname{mod} \Lambda$, consider the exact sequence $0 \to N \xrightarrow{g_0(N)} I_0(N) \xrightarrow{j} \Omega^{-1}N \to 0$ and a projective cover $f_0(I_0(N)): P_0(I_0(N)) \to I_0(N)$. Denote by h a unique morphism making the following exact diagram commutative.*

$$\begin{tikzcd} 0 \arrow[r] & \operatorname{Ker} jf_0(I_0(N)) \arrow[r, "i"] \arrow[d, "h"'] & P_0(I_0(N)) \arrow[r, "jf_0(I_0(N))"] \arrow[d, "f_0(I_0(N))"] & \Omega^{-1}N \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & N \arrow[r, "g_0(N)"] & I_0(N) \arrow[r, "j"] & \Omega^{-1}N \arrow[r] & 0. \end{tikzcd}$$

Then $h: \operatorname{Ker} jf_0(I_0(N)) \to N$ is a right Sub Λ-*approximation.*

Proof. Let $u: X \to N$ be a morphism with $X \in \operatorname{Sub} \Lambda$. By Lemma 4.4, $f_0(I_0(N))$ is a right Sub Λ-approximation of $I_0(N)$, so that $g_0(N)u = f_0(I_0(N))v$ for some $v: X \to P_0(I_0(N))$. Since i is a monomorphism, so is the morphism $\left[\begin{smallmatrix} h \\ i \end{smallmatrix}\right]: \operatorname{Ker} jf_0(I_0(N)) \to N \oplus P_0(I_0(N))$. Hence the left square of the diagram, which is a pushout, is also a pullback, so that there is a unique morphism $l: X \to \operatorname{Ker} jf_0(I_0(N))$ satisfying $u = hl$ and $v = il$. We have shown that h is a right Sub Λ-approximation of N. ∎

Note that, as follows from Proposition 4.7(b) to be stated and proved below, the morphism h from Corollary 4.5 is not a minimal right $\operatorname{Sub}\Lambda$-approximation of N, at least, if $N \in \operatorname{Sub}\Lambda$.

For the remainder of the paper, unless indicated otherwise, we assume that $\operatorname{Sub}\Lambda$ is closed under extensions.

We proceed with a more specific description of the Ext-injective modules in $\operatorname{Sub}\Lambda$.

COROLLARY 4.6. *Let $A \in \operatorname{Sub}\Lambda$ be an indecomposable module.*

(a) *A is splitting injective if and only if it is a direct summand of a projective cover $P_0(I)$ of some indecomposable injective Λ-module I.*

(b) *A is Ext-injective but not splitting injective if and only if A is a direct summand of ΩI for some indecomposable injective Λ-module I.*

Proof. (a) This follows from Lemma 4.4 and Theorem 4.2(a).

(b) This follows from Lemma 4.4 and Theorem 4.2(b). ∎

PROPOSITION 4.7. *Let A be an indecomposable module in $\operatorname{Sub}\Lambda$ that is not splitting injective.*

(a) *$I_0(A)$ and $\Omega^{-1}A$ have the same projective cover.*

(b) *$\Omega\Omega^{-1}A \cong \Omega I_0(A) \oplus A$.*

Proof. (a) Consider the exact commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, equal] & A \oplus P_0(\Omega^{-1}A) \arrow[r] \arrow[d, "g"] & P_0(\Omega^{-1}A) \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & A \arrow[r] & I_0(A) \arrow[r] \arrow[d] & \Omega^{-1}A \arrow[r] \arrow[d] & 0, \\
 & & 0 & 0 &
\end{tikzcd}$$

where the bottom row is an injective envelope and h is a projective cover. By Corollary 4.6(a), $P_0(I_0(A))$ is a splitting injective module and so is $P_0(\Omega^{-1}A)$ because $\Omega^{-1}A$ is a factor module of $I_0(A)$. Using Lemma 4.4, we conclude that $P_0(I_0(A))$ is a direct summand of $A \oplus P_0(\Omega^{-1}A)$. Since A is not a splitting injective module, $P_0(I_0(A))$ is a direct summand of $P_0(\Omega^{-1}A)$, whence we have $P_0(I_0(A)) \cong P_0(\Omega^{-1}A)$.

(b) Consider the exact commutative diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \Omega I_0(A) \arrow[r] \arrow[d, equal] & V \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Omega I_0(A) \arrow[r] & P_0(I_0(A)) \arrow[r] \arrow[d] & I_0(A) \arrow[r] \arrow[d] & 0 \\
 & & \Omega^{-1}A \arrow[r, equal] \arrow[d] & \Omega^{-1}A \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

where the middle row is a projective cover and the right column is an injective envelope. By (a), the middle column is a projective cover, so that $V \cong \Omega\Omega^{-1}A$. Since $\Omega I_0(A)$ is an Ext-injective module in $\operatorname{Sub}\Lambda$ according to Corollary 4.6(b), the top row splits. Thus $\Omega\Omega^{-1}A \cong \Omega I_0(A) \oplus A$. ∎

Denote by $\Omega(\mathcal{I})$ the subcategory of $\operatorname{mod}\Lambda$ consisting of the finite direct sums of indecomposable direct summands of ΩI, where I runs through the set of indecomposable injective Λ-modules. By Corollary 4.6(b), $\Omega(\mathcal{I})$ is the full additive subcategory of $\operatorname{Sub}\Lambda$ determined by the Ext-injective but not splitting injective modules. For all $N \in \operatorname{mod}\Lambda$, we have $N = N_{\Omega(\mathcal{I})} \oplus N'$, where $N' \in \Omega(\mathcal{I})$ and no indecomposable direct summand of $N_{\Omega(\mathcal{I})}$ is in $\Omega(\mathcal{I})$. Clearly, $N_{\Omega(\mathcal{I})}$ is uniquely determined up to isomorphism.

THEOREM 4.8. *Let* $0 \to A \to B \to C \to 0$ *be an almost split sequence in* $\operatorname{Sub}\Lambda$. *Then*:

(a) $C \cong \Omega \operatorname{Tr} \Omega \operatorname{D}A$.

(b) $\Omega^{-1}A \cong W \oplus I$, *where* I *is an injective module and* W *is an indecomposable non-injective module*.

(c) $\Omega W \cong \Omega \operatorname{D}\Omega \operatorname{Tr} C$.

(d) $A \cong (\Omega \operatorname{D}\Omega \operatorname{Tr} C)_{\Omega(\mathcal{I})}$.

(e) $A \cong \Omega \operatorname{D}\Omega \operatorname{Tr} C$ *if and only if* ΩW *is indecomposable*.

Proof. (a) It is an immediate consequence of Proposition 4.1(c) and the argument at the bottom of p. 409 of [BM], which works not only when Λ is 1-Gorenstein, but for all artin algebras Λ such that $\operatorname{Sub}\Lambda$ is closed under extensions.

(b) We begin with an observation that $\Omega \operatorname{D}A \cong A' \oplus Q$, where A' is an indecomposable non-projective Λ^{op}-module and Q is a projective Λ^{op}-

module. This follows from the facts that $C \cong \Omega \operatorname{Tr} \Omega \mathrm{D}A$ is an indecomposable non-projective Λ-module, Tr vanishes on projective modules, the operators Ω and Tr respect direct sums, and $\Omega \operatorname{Tr} \Omega \operatorname{Tr} Y \cong Y$ for all indecomposable non-projective $Y \in \operatorname{Sub} \Lambda^{op}$ as stated in Proposition 3.1 of [BM]. Therefore $\Omega^{-1}A \cong \mathrm{D}A' \oplus \mathrm{D}Q$, where $W = \mathrm{D}A'$ is indecomposable non-injective and $I = \mathrm{D}Q$ is injective.

(c) In view of (a), we have $\Omega\, \mathrm{D}\Omega \operatorname{Tr} C \cong \Omega\, \mathrm{D}\Omega \operatorname{Tr} \Omega \operatorname{Tr} \Omega\, \mathrm{D}A \cong \Omega\, \mathrm{D}A' = \Omega W$.

(d) Using (b) and Proposition 4.7(b), we obtain $\Omega\, \mathrm{D}\Omega\, \mathrm{D}A \cong \Omega\Omega^{-1}A \cong \Omega W \oplus \Omega I \cong \Omega I_0(A) \oplus A$. By the Krull−Remak−Schmidt theorem, $\Omega W \cong A \oplus Z$ with $Z \in \Omega(\,I\,)$ because A is an indecomposable non-Ext-injective module in Sub Λ and all modules in $\Omega(\,I\,)$ are Ext-injective according to Corollary 4.6(b). By (c), we have $A \cong (\Omega W)_{\Omega(\,I\,)} \cong (\Omega\, \mathrm{D}\Omega \operatorname{Tr} C)_{\Omega(\,I\,)}$.

(e) This follows from (c) and (d). ∎

Part (a) of Theorem 4.8 was obtained in [BM] in case Λ is 1-Gorenstein. Part (d) is a generalization of the formula $A \cong \Omega\, \mathrm{D}\Omega \operatorname{Tr} C$ from [BM]. We show that the latter formula does not work in general.

EXAMPLE 4.1. Consider the quiver $G = (v(G), a(G))$ with the set of vertices $v(G) = \{1, 2\}$ and the set of arrows $a(G) = \{\alpha, \beta\}$, where $\alpha : 1 \to 1$ and $\beta : 2 \to 1$. Fix an arbitrary field k and let $\Lambda = k[G, \rho(G)]$ be the path algebra of G over k modulo the ideal generated by the set of relations $\rho(G) = \{\alpha^3\}$.

Denote by e_1, e_2 the idempotents of Λ corresponding to the trivial paths on G at the vertices 1 and 2, respectively. Then $\{\Lambda e_1, \Lambda e_2\}$ is a complete set of pairwise non-isomorphic indecomposable projective Λ-modules. A direct verification shows that $\operatorname{Hom}_\Lambda(\Lambda e_2, \Lambda e_1) = 0$, Λe_1 is isomorphic to a submodule of Λe_2, and every non-zero endomorphism of Λe_2 is an isomorphism. It follows that Λe_2 is a unique up to isomorphism maximal projective Λ-module.

Denote by I_1 an injective envelope of the simple Λ-module S_1 associated with vertex 1, and by I_2, the simple injective module associated with vertex 2. A standard computation shows that the Λ-module I_1 corresponds to the following representation (V, f) of the quiver with relations (G, ρ) over k. The vector space V_1 associated to vertex 1 has a k-basis $\{u_1, u_2, u_3\}$; the vector space V_2 associated to vertex 2 has a k-basis $\{v_1, v_2, v_3\}$; the linear transformation f_α associated to arrow α is given by $f_\alpha(u_i) = u_{i+1}$ for $i = 1, 2$ and $f_\alpha(u_3) = 0$; the linear transformation f_β associated to arrow β is given by $f_\beta(v_i) = u_i$ for $i = 1, 2, 3$.

Since $\operatorname{Soc} \Lambda e_2 \cong S_1$, then I_1 is an injective envelope of Λe_2, and we obtain the exact sequence $0 \to \Lambda e_2 \to I_1 \to I_2 \oplus I_2 \to 0$. Therefore inj dim $\Lambda e_2 = 1$ and Sub Λ is closed under extensions according to Proposition 4.3(b).

It is easy to see that Λe_2 is a projective cover of I_2 and $\Lambda e_2 \oplus \Lambda e_2 \oplus \Lambda e_2$ is a projective cover of I_1, so that we obtain the exact sequences

$$0 \to \Lambda e_1 \to \Lambda e_2 \to I_2 \to 0 \tag{4.2}$$

and

$$0 \to \Lambda e_1 \oplus \Lambda e_1 \to \Lambda e_2 \oplus \Lambda e_2 \oplus \Lambda e_2 \to I_1 \to 0. \tag{4.3}$$

By Corollary 4.6(b), Λe_1 is a unique up to isomorphism indecomposable Ext-injective module that is not splitting injective. Since Λe_2 is a unique up to isomorphism maximal projective module, Proposition 4.3(a) implies that S_1 is an indecomposable non-Ext-injective module in Sub Λ.

We ask the reader to verify that we have the exact sequence $0 \to S_1 \to I_1 \to I_2 \oplus W \to 0$, where the Λ-module W corresponds to the following representation (W, g) of the quiver with relations (G, ρ) over k. The vector space W_1 associated to vertex 1 has a k-basis $\{u_1, u_2\}$; the vector space W_2 associated to vertex 2 has a k-basis $\{v_1, v_2\}$; the linear transformation g_α associated to arrow α is given by $g_\alpha(u_1) = u_2$ and $g_\alpha(u_2) = 0$; the linear transformation g_β associated to arrow β is given by $g_\beta(v_i) = u_i$ for $i = 1, 2$. It is easy to check that W is an indecomposable non-injective Λ-module.

It follows from (4.2) and (4.3) that $\Omega I_2 \cong \Lambda e_1$ and $\Omega I_1 \cong \Lambda e_1 \oplus \Lambda e_1$, so that $\Omega\Omega^{-1}S_1 \cong \Omega I_2 \oplus \Omega W \cong \Lambda e_1 \oplus \Omega W \cong \Omega I_1 \oplus S_1 \cong \Lambda e_1 \oplus \Lambda e_1 \oplus S_1$, using Proposition 4.7(b) and the fact that I_1 is an injective envelope of S_1. By the Krull−Remak−Schmidt theorem, $\Omega W \cong \Lambda e_1 \oplus S_1$. Since ΩW is decomposable, it follows from Theorem 4.8(e) that if $0 \to S_1 \to B \to C \to 0$ is an almost split sequence in Sub Λ, then $\Omega\, \mathrm{D}\Omega\, \mathrm{Tr}\, C \not\cong S_1$. Thus the formula for the left end-term of an almost split sequence in Sub Λ obtained in [BM] for a 1-Gorenstein artin algebra Λ does not work in general.

In view of Theorem 4.8(d), (e) and Example 4.1, it is natural to ask whether an artin algebra Λ is l-Gorenstein, provided $A \cong \Omega\, \mathrm{D}\Omega\, \mathrm{Tr}\, C$ for all almost split sequences $0 \to A \to B \to C \to 0$ in Sub Λ. To show that the answer is negative, we need an easy observation about artin algebras satisfying the following condition [JK].

(A) Every submodule of an indecomposable projective module is either projective or simple.

The following statement is true without the assumption that Sub Λ is closed under extensions.

LEMMA 4.9. *If* Λ *satisfies* (A), *then an indecomposable non-projective module in* Sub Λ *is simple*.

Proof. Consider a left minimal monomorphism $f: X \to \bigoplus_{j=1}^{t} P_j$, where $X \in \operatorname{Sub}\Lambda$ is indecomposable non-projective and P_j is indecomposable projective for all j. Let $\pi_l : \bigoplus_{j=1}^{t} P_j \to P_l$ be the projection. Then $\operatorname{Im} \pi_l f \neq 0$ is a simple module because Λ satisfies (A) and X is indecomposable non-projective. It follows that $X \cong \operatorname{Im} f$ is semisimple, hence, it must be simple. ∎

EXAMPLE 4.2. Consider the path algebra $\Lambda = k[G, \rho(G)]$ over an arbitrary field k of the quiver $G = (v(G), a(G))$ described in Example 4.1 modulo the ideal generated by the set of relations $\rho(G) = \{\alpha^2\}$. Using the same as in Example 4.1 notation for the simple, indecomposable projective, and indecomposable injective modules, we have that Λe_1 is a module of length 2 and $\mathbf{r}\Lambda e_2 \cong \Lambda e_1$. Therefore, Λ satisfies condition (A) and Lemma 4.9 implies that $\{S_1, \Lambda e_1, \Lambda e_2\}$ is a complete set of non-isomorphic indecomposable modules in Sub Λ.

As in Example 4.1, $\operatorname{Hom}_\Lambda(\Lambda e_2, \Lambda e_1) = 0$ and every non-zero endomorphism of Λe_2 is an isomorphism, so that Λe_2 is a unique up to isomorphism maximal projective Λ-module.

The indecomposable injective Λ-module I_1 corresponds to the following representation (V, f) of the quiver with relations (G, ρ) over k. V_1 has a k-basis $\{u_1, u_2\}$; V_2 has a k-basis $\{v_1, v_2\}$; f_α is given by $f_\alpha(u_1) = u_2$ and $f_\alpha(u_2) = 0$; f_β is given by $f_\beta(v_i) = u_i$ for $i = 1, 2$.

Since $\operatorname{Soc} \Lambda e_2 \cong S_1$, then I_1 is an injective envelope of Λe_2, and we obtain the exact sequence $0 \to \Lambda e_2 \to I_1 \to I_2 \to 0$. Therefore, Λ is not 1-Gorenstein but $\operatorname{inj\,dim} \Lambda e_2 = 1$, whence Sub Λ is closed under extensions.

It is easy to see that Λe_2 is a projective cover of I_2 and $\Lambda e_2 \oplus \Lambda e_2$ is a projective cover of I_1, so that we obtain the exact sequences

$$0 \to \Lambda e_1 \to \Lambda e_2 \to I_2 \to 0 \tag{4.4}$$

and

$$0 \to \Lambda e_1 \to \Lambda e_2 \oplus \Lambda e_2 \to I_1 \to 0. \tag{4.5}$$

By Corollary 4.6(b), Λe_1 is a unique up to isomorphism indecomposable Ext-injective module that is not splitting injective. It follows that S_1 is a unique up to isomorphism indecomposable non-Ext-injective module in Sub Λ.

Embedding S_1 in its injective envelope I_1, we get the exact sequence $0 \to S_1 \to I_1 \to I_2 \oplus W \to 0$, where the Λ-module W corresponds to the following representation (W, g) of the quiver with relations (G, ρ) over

$k : W_1 = W_2 = k$, $g_\alpha = 0$, and $g_\beta = 1_k$. Clearly, W is an indecomposable non-injective Λ-module, and it is easy to see that $\Omega W \cong S_1$. By Theorem 4.8(e), for the almost split sequence $0 \to S_1 \to B \to C \to 0$ in $\operatorname{Sub}\Lambda$, we have $\Omega\, \mathrm{D}\Omega\, \mathrm{Tr}\, C \cong S_1$.

Since S_1 is a unique indecomposable non-Ext-injective module in $\operatorname{Sub}\Lambda$, we have $A \cong \Omega\, \mathrm{D}\Omega\, \mathrm{Tr}\, C$ for all almost split sequences $0 \to A \to B \to C \to 0$ in $\operatorname{Sub}\Lambda$. However, Λ is not a 1-Gorenstein artin algebra.

It would be interesting to describe the class of artin algebras for which $\operatorname{Sub}\Lambda$ is closed under extensions and $A \cong \Omega\, \mathrm{D}\Omega\, \mathrm{Tr}\, C$ for all almost split sequences $0 \to A \to B \to C \to 0$ in $\operatorname{Sub}\Lambda$. Example 4.2 shows that this class is strictly larger than the class of 1-Gorenstein algebras.

REFERENCES

[A] M. Auslander, Functors and morphisms determined by objects, *in* "Representation Theory of Algebras," Proceedings of the Philadelphia Conference, Lecture Notes in Pure and Applied Mathematics, Vol. 37, Dekker, New York, 1978.

[AR91a] M. Auslander and I. Reiten, Applications of contravariantly finite subcategories, *Adv. Math.* **86** (1991), 111–152.

[AR91b] M. Auslander and I. Reiten, Cohen–Macaulay and Gorenstein Artin algebras, *in* "Representation Theory of Finite Groups and Finite-Dimensional Algebras," Proceedings of the Conference at the University of Bielefeld from May 15–17, 1991, and 7 Survey Articles on Topics of Representation Theory, Progress in Mathematics, Vol. 95, Birkhauser Verlag, Basel, 1991.

[AR92] M. Auslander and I. Reiten, Homologically finite subcategories, *in* "Representations of Algebras and Related Topics," London Mathematical Society Lecture Series, Vol. 168, Cambridge Univ. Press, Cambridge, UK, 1992.

[ARS] M. Auslander, I. Reiten, and S. O. Smalø, "Representation Theory of Artin Algebras," Cambridge Studies in Advanced Mathematics, Vol. 36, Cambridge Univ. Press, New York, 1994.

[AS80] M. Auslander and S. O. Smalø, Preprojective modules over artin algebras, *J. Algebra* **66** (1980), 61–122.

[AS81a] M. Auslander and S. O. Smalø, Almost split sequences in subcategories, *J. Algebra* **69** (1981), 426–454.

[AS81b] M. Auslander and S. O. Smalø, Addendum to "Almost split sequences in subcategories," *J. Algebra* **71** (1981), 592–594.

[BM] R. Bautista and R. Martinez, Representations of partially ordered sets and 1-Gorenstein artin algebras, *in* "Proceedings, Conference on Ring Theory, Antwerp, 1978," pp. 385–433, Dekker, New York, 1979.

[H] M. Hoshino, On splitting torsion theories induced by tilting modules, *Comm. Algebra* **11**, No. 4 (1983), 427–439.

[JK] S. Jagadeeshan and M. Kleiner, Structure of projectively stable artinian rings, *J. Algebra* (1997), in press.

[M] S. MacLane, "Homology," Springer-Verlag, New York, 1967.

[W] T. Wakamatsu, Stable equivalence of self-injective algebras and a generalization of tilting modules, *J. Algebra* **134** (1990), 298–325.